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SINGULAR SEMI-LINEAR EQUATIONS IN $L^1(\mathbf{R})^\dagger$

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ABSTRACT

Let g be a positive continuous function on $\mathbf{R}$ which tends to zero at $-\infty$ and which is not integrable over $\mathbf{R}$. The boundary-value problem $-u''+g(u)=f$, $u'(\pm\infty)=0$, is considered for $f\in L^1(\mathbf{R})$. We show that this problem can have a solution if and only if g is integrable at $-\infty$ and if this is so then the problem is solvable precisely when $\int_{-\infty}^{\infty} f(t)dt>0$. Some extensions of this result are also given.

In [2] M. G. Crandall and L. C. Evans show that the singular semi-linear problem

$$(*)\qquad \begin{cases} -u''(x)+\beta(u(x))=f(x), & -\infty<x<\infty \\ u'(\pm\infty)=0 \\ u''\in L^1(\mathbf{R}) \end{cases}$$

has a solution for each $f\in L^1(\mathbf{R})$ with $\int_{\mathbf{R}} f>0$ if (and only if) β is integrable at $-\infty$. Here β is a given positive monotone increasing continuous function on $\mathbf{R}$. In fact, they discuss the more general situation when β is a maximal monotone graph. In this paper we consider several extensions of the problem (*) and provide another technique for proving that these equations have a solution. In particular, we recover the result of Crandall and Evans by different means.

THEOREM 1. *Let g be a positive continuous function on* $\mathbf{R}$ *with*

$$\lim_{t\to-\infty} g(t)=0,\ \int_{-\infty}^{\infty} g(s)ds \quad \textit{divergent.}$$

Let $L^1_+=\{f\in L^1(\mathbf{R}):\int_{\mathbf{R}} f>0\}$; for $f\in L^1_+$ consider the problem

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$$(1) \qquad \begin{cases} -u''(x)+g(u(x))=f(x), & -\infty<x<\infty \\ u''\in L^1(\mathbf{R}) \\ u'(\pm\infty)=0. \end{cases}$$

The following are equivalent:

(a) (1) *has a solution for all* $f \in L^1_+$,

(b) (1) *has a solution for some* $f \in L^1_+$,

(c) g *is integrable at* $-\infty$.

PROOF. (a) implies (b) is trivial. To see that (b) implies (c) suppose there is a function u with $u''\in L^1$, $u'(\pm\infty)=0$, and

$$(2) \qquad -u''+g(u)=f$$

for some $f\in L^1$. Then $u'\in L^\infty$ and u tends to $-\infty$ at both $\pm\infty$ for the following reason. Suppose there is a sequence $x_n\to\infty$ with $\lim_n u(x_n)=L>-\infty$. Let $\{y_n\}$ be any other sequence of real numbers tending to $+\infty$. Then from (2) we get

$$-\frac{1}{2}(u'(y_n))^2+\frac{1}{2}(u'(x_n))^2+H(u(y_n))-H(u(x_n))=\int_{x_n}^{y_n} fu'$$

where

$$H(t)=\int_0^t g(s)ds.$$

Hence, $\lim_n H(u(y_n))$ exists and equals $H(L)$. Thus, $H(u(t))$ has a limit at ∞ which implies that u has limit L at ∞ since H is strictly monotone. But then $g(u(t))$ tends to $g(L)>0$ as $t\to\infty$ which contradicts the fact that $g(u(t))$ is in $L^1(\mathbf{R})$. An identical argument shows u tends to $-\infty$ at $-\infty$. With H as above we also have

$$\frac{1}{2}[(u'(y))^2-(u'(0))^2]+H(u(0))-H(u(y))=\int_y^0 fu'$$

for each y, $y<0$. Thus, $H(u(y))$ has a finite limit as $y\to-\infty$. Since $u(y)\to-\infty$ as $y\to-\infty$ we find that $H(s)$ has a finite limit as $s\to-\infty$ implying that g is integrable at $-\infty$. The proof that (c) implies (a) is the most difficult. The first step is to show that the set of those $f\in L^1_+$ for which (1) is solvable is closed in L^1_+; the second step is then obviously to show that the set of those $f\in L^1_+$ for which (1) is solvable is dense in L^1_+. To prove the first assertion, let $f_n\to f$ in $L^1(\mathbf{R})$, with f, $f_n\in L^1_+$. Let u_n satisfy

$$-u_n'' + g(u_n) = f_n, \tag{3a}$$

$$u_n'(\pm\infty) = 0, \tag{3b}$$

$$u_n'' \in L^1(\mathbf{R}). \tag{3c}$$

Integrate both sides of (3a) from $-\infty$ to x and then from x to $+\infty$ and use the fact that $g \geqq 0$. This gives $|u_n'(x)| \leqq \|f\|_1 + 1$ for all large n and hence

$$\|u_n'\|_{L^\infty(\mathbf{R})} \leqq A, \qquad n = 1, 2, \cdots. \tag{4}$$

This in turn implies that $\{u_n\}$ is equicontinuous. We may assume, therefore, that $\{u_n\}$ converges uniformly on compact subsets of $\mathbf{R}$ to either $+\infty$, or $-\infty$, or to a continuous function u. Set

$$G(x) = \int_{-\infty}^{x} g(s)ds.$$

For any $x \in \mathbf{R}$ and any n we have

$$\begin{aligned} G(u_n(x)) &= \int_{-\infty}^{x} g(u_n(t))u_n'(t)dt \\ &= \frac{1}{2}[u_n'(x)]^2 + \int_{-\infty}^{x} f_n u_n' \\ &\leqq A_1. \end{aligned}$$

Hence, $u_n(x) \leqq C$ for all n and all x. Thus, it is obviously impossible that $\{u_n\}$ tends to $+\infty$. Suppose that $\{u_n\}$ tends to $-\infty$ uniformly on compact subsets of $\mathbf{R}$. Again we have

$$-\frac{1}{2}(u_n'(x))^2 + G(u_n(x)) = \int_{-\infty}^{x} f_n(t)u_n'(t)dt \tag{5}$$

and hence

$$0 = \int_{-\infty}^{\infty} f_n u_n'. \tag{6}$$

We may assume that $\{u_n'\}$ converges weak-* in $L^\infty(\mathbf{R})$ to a function p and also that $\{u_n'(0)\}$ converges. Integrating (3a) from 0 to x we see that $u_n'(x)$ converges pointwise to $p(x)$ on $\mathbf{R}$. Hence, (5) and (6) yield

$$-\frac{1}{2}(p(x))^2=\int_{-\infty}^{x} fp$$

and

$$0=\int_{-\infty}^{\infty} fp.$$

Hence, p has a limit of 0 at both $+\infty$ and $-\infty$. Again from (3a) we obtain

$$u_n'(y)-u_n'(x)+\int_y^x g(u_n(t))dt=\int_y^x f_n(t)dt$$

so that

$$p(y)-p(x)=\int_y^x f(t)dt.$$

Now let $y\to-\infty$ and $x\to+\infty$; we find

$$0<\int_{-\infty}^{\infty} f=p(-\infty)-p(+\infty)=0,$$

a contradiction. Note that this argument is dependent on g in only a minor way. In particular, if $\{g_n\}$ is a sequence of positive continuous functions converging uniformly on compact subsets to a positive continuous function g which tends to 0 at $-\infty$ and which lies in $L^1(-\infty,0]$ but not in $L^1(\mathbf{R})$ and if, say, $\{g_n\}$ increases to g on $(-\infty,\infty)$, then the functions v_n which satisfy

$$-v_n''+g_n(v_n)=f,\qquad v_n'(\pm\infty)=0,\qquad f\in L_+^1$$

are equicontinuous and uniformly bounded on compact subsets of $\mathbf{R}$. We shall make use of this later on.

Returning to the functions $\{f_n\}$ and $\{u_n\}$ we see that $\{u_n\}$ converges uniformly on compact subsets of $\mathbf{R}$ to a continuous function u. We clearly have $u_n''\to u''$ in L_{loc}^1 so that u satisfies

$$(7)\qquad -u''+g(u)=f\quad\text{on}\quad \mathbf{R}.$$

Fatou's lemma implies $g(u)$ is in $L^1(\mathbf{R})$ and hence $u''\in L^1(\mathbf{R})$; thus u' has limits at both $\pm\infty$ and u tends to $-\infty$ at both $\pm\infty$ as in the implication (b) implies (c). From (5) and (6) we get

$$-\frac{1}{2}(u'(x))^2+G(u(x))=\int_{-\infty}^{x} fu'$$

and

$$0=\int_{-\infty}^{\infty} fu'.$$

Hence, u' tends to 0 at both $\pm\infty$, so that u is a solution of (1). Note also that

$$\int_{-\infty}^{\infty} |u''+f| = \int_{-\infty}^{\infty} (u''+f) = \int_{-\infty}^{\infty} f$$
$$\leqq \|f\|_1$$

and hence

$$\|u''\|_1 \leqq 2\|f\|_1. \tag{8}$$

The second assertion, that there is a dense set of $f \in L_+^1$ for which (1) is solvable, will be proved in the following way. Let f be a continuous function on $\mathbf{R}$ in L_+^1 with support in the interval $I=[a,b]$. We shall show (1) is solvable for this f. We assume temporarily that g is C^1 on $\mathbf{R}$.

We shall need the following Proposition.

PROPOSITION. *Let $a<b$ and let g be a positive C^1 function on* $\mathbf{R}$ *which is integrable at $-\infty$ and bounded at $+\infty$; set*

$$G(x)=\int_{-\infty}^{x} g(s)ds.$$

Then for each α, β the initial value problem

$$\begin{aligned} &-v''(x)+g(v(x))=f(x), \qquad a<x<b, \qquad f\in L^2(a,b)\\ &v(a)=\alpha, \qquad v'(a)=\beta \end{aligned} \tag{9}$$

has a unique solution. If $\alpha_n \to \alpha$ and $\beta_n \to \beta$ and if v_n is the solution of (9) *for (α_n, β_n), then v_n converges uniformly to the solution v of* (9) *for (α, β). Finally, the family $\{v_{\alpha\beta}\}$ of solutions of* (9) *corresponding to the initial values $\{(a,\beta): -\infty<\alpha\leqq\alpha_0, |\beta|\leqq M\}$ is equicontinuous on $[a,b]$.*

PROOF. Once the equicontinuity is established the existence and uniqueness follow from standard results; see [1], Chapter 1. To obtain the equicontinuity assertion (from which the second assertion also follows), we multiply the top equation in (9) by v' and integrate to obtain

$$-\frac{1}{2}(v'(x))^2+G(v(x))+\frac{1}{2}\beta^2-G(\alpha)=\int_a^x fv'$$

so that if x_0 is chosen with $|v'(x_0)| = \|v'\|_\infty$ we have

$$\begin{aligned}\|v'\|_\infty^2 &\leqq \beta^2 + 2G(\alpha) + 2G(v(x_0)) + A\|v'\|_\infty \\ &\leqq \beta^2 + 2G(\alpha) + 2G(\alpha + (b-a)\|v'\|_\infty) + A\|v'\|_\infty \\ &\leqq \beta^2 + 2G(\alpha) + A_0 + A_1(\alpha + (b-a)\|v'\|_\infty) + A\|v'\|_\infty\end{aligned}$$

for some constants A_0, A_1 depending only on g. Hence, $\|v'\|_\infty$ is bounded for $|\beta| \leqq M$ and $-\infty < \alpha \leqq \alpha_0$.

Conclusion of Proof of Theorem 1. Let f be a continuous function in L^1_+ with support in the interval (a, b). We shall show that (1) is solvable for this f. First, on $(-\infty, a]$ we show that the equation

$$g(u(x)) = u''(x) \tag{10}$$

$$u(a) = c_1, \qquad u'(-\infty) = 0$$

has a solution. Let v be the function with

$$\begin{aligned} &v'(t) = (2G(t))^{-1/2}, \qquad -\infty < t < c_1 \\ &v(c_1) = a \end{aligned}$$

where

$$G(x) = \int_{-\infty}^{x} g(s)ds.$$

Then v is increasing and has range $(-\infty, a]$. Let u be the inverse of v on $(-\infty, a]$, $u(v(t)) = t$. Thus

$$u(a) = c_1$$

and

$$u'(x) = 1/v'(t) = (2G(t))^{1/2}$$

or

$$u'(x) = (2G(u(x)))^{1/2}. \tag{11}$$

If we differentiate both sides of (11) we see that u satisfies (10). Similarly, there is a solution of

$$u''(x) = g(u(x)) \qquad b < x < \infty$$

$$u(b) = c_2, \qquad u'(\infty) = 0$$

which satisfies

$$u'(x) = -(2G(u(x)))^{1/2}, \qquad b < x < \infty.$$

Hence, to finish the proof of the theorem we need only show that there is a solution v of the equation

$$(12) \qquad -v'' + g(v) = f \quad \text{on} \quad (a, b)$$

with

$$(13) \qquad \begin{aligned} &\text{(a)} \quad v'(a) = (2G(v(a)))^{1/2}, \\ &\text{(b)} \quad v'(b) = -(2G(v(b)))^{1/2}. \end{aligned}$$

Let v_t be the solution of (12) with $v(a) = t$ and $v'(a) = (2G(t))^{1/2}$ assured by the Proposition. (We temporarily assume that g is bounded at $+\infty$ if, in fact, it is not.) Then

$$\begin{aligned} v_t'(b) &= v_t'(a) + \int_a^b v_t''(s)ds \\ &= (2G(t))^{1/2} + \int_a^b g(v_t(s))ds - \rho \end{aligned}$$

where $\rho = \int_a^b f(t)dt > 0$. To show that t may be chosen with $v_t'(b) = -(2G(v_t(b)))^{1/2}$ we consider

$$l(t) = (2G(t))^{1/2} + (2G(v_t(b)))^{1/2} + \int_a^b g(v_t(s))ds - \rho.$$

The Proposition implies l is continuous. We have

$$l(t) \geq -\rho + (2G(t))^{1/2}.$$

Since G is unbounded, there are values of t with $l(t) > 0$. Next let $t \downarrow -\infty$; by the equicontinuity of the functions $\{v_t\}$ we must have $v_t \to -\infty$ uniformly on $[a, b]$ so that $l(t) \to -\rho < 0$; hence, there is a t_0 at which $l(t_0) = 0$, and thus (12) is solvable with the boundary conditions (13).

We have now shown that (1) is solvable for all $f \in L_+^1$ under the assumption

$$(14) \qquad g \in C^1(\mathbf{R}) \cap L^\infty(\mathbf{R}), \qquad g \notin L^1(\mathbf{R}).$$

If g is merely positive and continuous on $\mathbf{R}$ with $g \in L^1(-\infty, 0)$, $g \notin L^1(\mathbf{R})$, then there is a sequence $\{g_n\}$ of positive functions satisfying (14) which converge uniformly on compact subsets of $\mathbf{R}$ to g and which also increase to g on $(-\infty, \infty)$.

The comments made earlier show that the solutions $\{u_n\}$ of (1) with g_n in place of g converge to a solution of (1) for g. This completes the proof of Theorem 1.

REMARK. The condition $g \notin L^1(\mathbf{R})$ is necessary as well as sufficient in order that Theorem 1 be valid. For suppose $g \in L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$; then the function G is bounded. If f is supported on $[-1, 1]$ and if (1) has a solution for f, then (13) must hold with u in place of v so that

$$0 = \sqrt{2G(u(-1))} + \sqrt{2G(u(1))} + \int_{-1}^{1} g(u(s))ds - \int_{-1}^{1} f(s)ds.$$

The first three terms of this expression are bounded, independent of u, and hence the integral of f over $\mathbf{R}$ can not exceed some fixed number depending only on g.

THEOREM 2. *Let g be a positive continuous function on $\mathbf{R}$ with*

(15)
$$\lim_{t \to -\infty} g(t) = 0, \qquad g \notin L^1(\mathbf{R}).$$

Let $B(x)$ be a positive absolutely continuous function on $\mathbf{R}$ with $B' \in L^1(\mathbf{R})$ and B bounded away from zero. For $f \in L^1_+$ consider the equation

(16)
$$\begin{cases} -u''(x) + B(x)g(u(x)) = f(x), & -\infty < x < \infty \\ u'' \in L^1(\mathbf{R}), & \\ u'(\pm\infty) = 0. & \end{cases}$$

Then (16) has a solution for each $f \in L^1_+$ if and only if g is integrable at $-\infty$.

PROOF. If (16) is solvable for some $f \in L^1_+$ with support in $[-1, 1]$ then $u' > 0$ on $(-\infty, -1]$ and $u' < 0$ on $[1, \infty)$. It now follows very much as in Theorem 1 that u tends to $-\infty$ at $\pm\infty$ and that g is integrable at $-\infty$.

To show the sufficiency of the condition that g be integrable at $-\infty$ we first show that the equations

(17)
$$\begin{cases} u''(x) = B(x)g(u(x)), & |x| \geq a > 0 \\ u(-a) = c_1, \quad u(a) = c_2 & \\ u'(\pm\infty) = 0 & \end{cases}$$

have a solution. As in the proof of Theorem 1, the solution u must be monotone increasing for $-\infty < x < -a$ and monotone decreasing on (a, ∞); we shall only consider the details for the case $-\infty < x < -a$, the other case being entirely similar. We wish to find a continuous function v with

$$(18)\qquad \begin{aligned} &v'(t) = \left(2\int_{-\infty}^{t} B(v(s))g(s)ds\right)^{-1/2}, \qquad -\infty < t < c_1 \\ &v(c_1) = -a. \end{aligned}$$

If such a v exists, then the inverse function u of v will satisfy

$$u'(x) = \left(2\int_{-\infty}^{x} B(r)g(u(r))u'(r)dr\right)^{1/2},$$

$$u(-a) = c_1$$

and hence u will satisfy (17). To see that (18) has a solution let b_1 and b_2 be positive numbers with $b_1 \leqq B(s) \leqq b_2$ for all s and let ξ_N be the function defined by

$$\xi_N(t) = \left(2\int_{-N}^{t} g(s)ds\right)^{-1/2}, \qquad -N \leqq t \leqq c_1.$$

Let $\Omega_N = \{w \in C(-N, c_1): (2b_2)^{-1/2}\xi_N(t) \leqq w(t) \leqq (2b_1)^{-1/2}\xi_N(t)$ for all $t \in [-N, c_1]\}$ and let T map Ω_N into Ω_N by

$$(Tw)(x) = \left(2\int_{-N}^{x} B(\tilde{w}(s))g(s)ds\right)^{-1/2}$$

where

$$\tilde{w}'(t) = w(t), \qquad \tilde{w}(c_1) = -a.$$

Clearly $Tw \in \Omega_N$; if $\{w_n\}$ is a bounded sequence in Ω_N, then $\{\tilde{w}_N\}$ is equicontinuous and uniformly bounded. Thus, T is a compact mapping and so has a fixed point w_N which must satisfy

$$w_N(x) = \left(2\int_{-N}^{x} B(\tilde{w}_N(s))g(s)ds\right)^{-1/2}, \qquad -N \leqq x \leqq c_1.$$

The functions $\{\tilde{w}_N\}$ are equicontinuous and uniformly bounded on compact subsets of $(-\infty, c_1]$ and so a subsequence, again denoted by $\{\tilde{w}_N\}$, converges uniformly on compact subsets of $(-\infty, c_1]$ to a function $\tilde{w}_0$. But we also see that

$$\int_{-N}^{x} B(\tilde{w}_N(s))g(s)ds \to \int_{-\infty}^{x} B(\tilde{w}_0(s))g(s)ds$$

uniformly on compact subsets of $(-\infty, c_1]$. Hence, $w_N \to w_0$ uniformly on compacta; setting $v = \tilde{w}_0$ we see that v satisfies (18).

The remainder of the proof of Theorem 2 is like that of Theorem 1; the condition that $B' \in L^1(\mathbf{R})$ is used to prove that the sequence $\{u_n\}$ can not go to $-\infty$.

COROLLARY 3. *Let* $a(x) \in L^1(\mathbf{R})$, $f \in L^1(\mathbf{R})$, *and let* g *be a positive continuous function satisfying* (15). *Consider the equation*

$$(19)\quad \begin{cases} \text{(i)} & -u''(x) + a(x)u'(x) + g(u(x)) = f(x), \qquad -\infty < x < \infty \\ \text{(ii)} & u'' \in L^1(\mathbf{R}) \\ \text{(iii)} & u'(\pm\infty) = 0. \end{cases}$$

Let g *be integrable at* $-\infty$ *and set* $w(x) = \exp[-\int_0^x a(s)ds]$. *A necessary and sufficient condition that* (19) *be solvable is that*

$$(20)\qquad \int_{\mathbf{R}} f(x)w(x)dx > 0.$$

If (19) *is solvable for all* $f \in L^1$ *satisfying* (20), *then* g *is integrable at* $-\infty$.

PROOF. Let $x = H(y)$ where H is the inverse of the function I defined by

$$I'(x) = 1/w(x),$$
$$I(0) = 0.$$

Then both H and I are 1-1 monotone increasing functions mapping $\mathbf{R}$ onto $\mathbf{R}$ and the substitution $v(y) = u(H(y))$ reduces (19) to

$$(21)\quad \begin{cases} -v''(y) + (H'(y))^2 g(v(y)) = (H'(y))^2 f(H(y)), \\ v'' \in L^1, \qquad v'(\pm\infty) = 0 \end{cases}$$

which has a solution according to Theorem 2 precisely when

$$0 < \int_{-\infty}^{\infty} (H'(y))^2 f(H(y))dy$$
$$= \int_{-\infty}^{\infty} f(x)w(x)dx.$$

REMARK. Let β be a maximal monotone graph lying in the upper half-plane; that is, $\beta(x)$ is a subset of $\{y > 0\}$ for each $x \in \mathbf{R}$. Let $\beta^0(x) = \min\{y : y \in \beta(x)\}$. The result of Crandall and Evans is that if

$$\int_{-\infty}^{a} \beta^0(x)dx < \infty$$

for some $a \in D(\beta)$, then the equation

$$-u''(x)+\beta(u(x))\ni f(x), \qquad u'(\pm\infty)=0, \qquad f\in L^1_+ \tag{22}$$

is solvable. This result also follows from Theorem 1 in the following way. Let $\{\beta_n\}$ be a sequence of positive continuous monotone increasing functions which increase to β^0 on $D(\beta)$ and which increase to $+\infty$ off $D(\beta)$. The solutions $\{u_n\}$ of (1) with β_n in place of g then decrease on $\mathbf{R}$ to a solution u of (22).

A final result related to Theorem 1 is presented below.

THEOREM 4. *Let g be a positive continuous function on* $\mathbf{R}$ *satisfying* (15). *For* $f\in L^1(\mathbf{R})$ *consider the equation*

$$\begin{cases} u''(x)+g(u(x))=f(x), & -\infty<x<\infty \\ u''\in L^1(\mathbf{R}) \\ u'(-\infty)=\xi_1, \qquad u'(+\infty)=\xi_2 \end{cases} \tag{23}$$

where

$$\int_{-\infty}^{\infty} f(x)dx=\rho>\xi_2-\xi_1. \tag{24}$$

(a) *Suppose g is integrable at* $-\infty$. *If* (23) *has a solution for some f with compact support* (*which necessarily satisfies* (24)) *then* $\xi_1>0>\xi_2$. *If* (23) *has a solution for* $f\equiv 0$, *then* $\xi_1=-\xi_2$.

(b) *If g is integrable at* $-\infty$ *and if* $\xi_1>0>\xi_2$, *then* (23) *has a solution for all f with*

$$\xi_2-\xi_1<\rho=\int_{-\infty}^{\infty} f(x)dx \leqq \min\{\xi_2, -\xi_1\}. \tag{25}$$

(c) *If* (23) *has a solution for some f satisfying* (24), *then g is integrable at* $-\infty$.

PROOF. (a). If f has support in $[a, b]$, then $u''(x)<0$ for $x<a$ and $x>b$. If $u'(-\infty)\leqq 0$, then $u'<0$ on $(-\infty, a)$ and hence u is decreasing on $(-\infty, a)$. However, u must tend to $-\infty$ at both $-\infty$ and $+\infty$ if u is a solution of (23) and thus u can not decrease on $(-\infty, a)$. Likewise, $u'(+\infty)$ must be negative. Further, if $u''+g(u)\equiv 0$, then

$$(u'(x))^2+2G(u(x))\equiv const. \text{ on } (-\infty,\infty)$$

which clearly implies that $\xi_1=-\xi_2$.

(c) is proved exactly as in Theorem 1.

(b) is the most difficult of the assertions. First, exactly as in Theorem 1, it can be shown that the set of those f satisfying (24) for which (23) is solvable is closed in $L^1(\mathbf{R})$. Next, we show that if f has compact support, say in (a, b), and if f satisfies

$$\xi_2 - \xi_1 < \rho = \int_{-\infty}^{\infty} f(x)dx < \min(-\xi_1, \xi_2) \tag{25$'$}$$

then (23) has a solution. The key to this, as in Theorem 1, is to show two things: first that the equations

$$\begin{aligned} &u''(x) + g(u(x)) = 0, \qquad x \notin [a, b] \\ &u'(-\infty) = \xi_1, \qquad u'(\infty) = \xi_2 \end{aligned} \tag{26}$$

have a solution which necessarily satisfies

$$\begin{aligned} u'(a) &= (\xi_1^2 - 2G(u(a)))^{1/2}, \\ u'(b) &= -(\xi_2^2 - 2G(u(b)))^{1/2} \end{aligned} \tag{27}$$

and second that the equation

$$u''(x) + g(u(x)) = f(x), \qquad a \leqq x \leqq b \tag{28}$$

is solvable subject to the non-linear boundary conditions (27). Both these assertions are proved as the similar statements are in the proof of Theorem 1.

REMARK. The upper bound in (25) is not completely satisfactory; however, the situation for (23) is more involved than that of (1) as (a) shows.

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